ALGEBRAIC CURVES
SOLUTIONS SHEET 11

Exercise 1. Let F be a projective plane curve.

(1) Let $P \in \mathbb{P}^2_k$. Show that P is a multiple point of F if, and only if, $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$.
(2) Suppose F is irreducible. Show that F has finitely many multiple points.
(3) Suppose F is nonsingular. Show that F is irreducible.

Now assume that F is irreducible of degree n.

(4) Show that F has at most $\frac{1}{2}n(n-1)$ multiple points. (Hint: combine Bezout's theorem with previous questions.)

Solution 1.

(1) Wlog, let $P = [1 : 0 : 0]$. P is a simple point $\iff$ $F_*(Y, Z)$ has a simple point at $(0, 0)$. Denote $F_1(Y, Z)$ its degree 1 part. Then $F_1(Y, Z) \neq 0 \iff (F_*)_Y(0, 0) \neq 0$ or $(F_*)_Z(0, 0) \neq 0$. Moreover, $(F_*)_Y = (F_Y)_*$, $(F_*)_Z = (F_Z)_*$. Note that by the formula
$$dF = XF_X + Y F_Y + ZF_Z$$
$F(P) = 0$ and $F_Y(P) = F_Z(P) = 0$ together with $P \in \{X \neq 0\}$ implies $F_X(P) = 0$.

(2) The proof is completely analogous to Exercise 4.4 on Sheet 8 (and you could deduce the version here from the version there by working in a standard affine cover). By point (1), the locus of multiple points is given by $M = V(F, F_X, F_Y, F_Z)$. If we assume by contradiction that this is infinite, then it must have dimension at least 1 by Exercise 2 on Sheet 7. Then, by Exercise 1.4 on Sheet 7, we obtain $M = V(F)$, and thus in particular $F_X, F_Y, F_Z \in I_p(F)$. As F is irreducible, we have $I_p(F) = (F)$, and as the degree of F_X, F_Y, F_Z is strictly smaller than the degree of F, we hence obtain that all the partial derivatives are equal to 0. In characteristic 0, this would imply that F is constant, contradiction. If the characteristic of k is $p > 0$, then we obtain that $F = G(X^p, Y^p, Z^p)$ for some homogeneous form $G \in k[X, Y, Z]$. But then, if we define $G^{1/p} := \sum_{i,j,l} G^{1/p}_{i,j,l} X^i Y^j Z^l$, we obtain $F = (G^{1/p})^p$, contradicting the fact that F is irreducible.

(3) Suppose F reducible, $F = F_1F_2$. Then by Bezout $V(F_1)$ and $V(F_2)$ intersects at least in one point P. As we have
$$F_X = F_1(F_2)_X + (F_1)_X F_2,$$

and $F_1(P) = F_2(P) = 0$, it follows that $F_X(P) = 0$. The same holds also for the other partial derivatives, and hence we obtain that P is a multiple point, contradiction. Hence F must be irreducible.

(4) By the proof of point (2), we must have that at least one of the partial derivatives of F doesn't vanish; without loss of generality we assume $F_X \neq 0$. Let P be a multiple point of F. We distinguish two cases:
- $P = [1:0:0]$. As $P \in F$, we can write $F = X^{n-1}F_1(Y,Z) + \cdots + F_n(Y,Z)$. Then $F_X = (n-1)X^{n-2}F_1(Y,Z) + \cdots + F_{n-1}(Y,Z)$. In particular, we have $m_P(F_X) \geq m_P(F)$ (with equality if and only if $\operatorname{char} k \nmid n-1$).
- $P \in \{Y \neq 0\} \cup \{Z \neq 0\}$, wlog $P \in \{Y \neq 0\}$. Let F_* be the dehomogenization with respect to Y, and write $P = [x_P : 1 : z_P]$. Then if $m = m_P(F)$, we have

$$F_*(x_P + X, z_P + Z) = F_m(X,Z) + (\text{higher order terms})$$

for some homogeneous m–form F_m. Then notice that

$$(F_X)_*(x_P + X, z_P + Z) = (F_*)_X(x_P + X, z_P + Z) = (F_m)_X(X,Z) + \cdots$$

so as $(F_m)_X$ is an $(m-1)$-form (or 0), we obtain that $m_P(F_X) \geq m - 1 = m_P(F) - 1$.

In conclusion, we have $m_P(F_X) \geq m_P(F) - 1$ for all $P \in F$. Using Bézout's theorem for F and F_X, we have

$$\begin{aligned} n(n-1) \geq \deg F \cdot \deg F_X &= \sum_P I(P, F \cap F_X) \geq \sum_P m_P(F) m_P(F_X) \\ &\geq \sum_P m_P(F)(m_P(F)-1) \geq \sum_{P:\ m_P(F)>1} 2 \end{aligned}$$

Hence the number of multiple points is bounded above by $n(n-1)/2$.

Exercise 2. Let F be an affine plane curve.

(1) Show that a line L is tangent to F at P if, and only if, $I(P, F \cap L) > m_P(F)$. This justifies the definition of tangent lines for projective plane curves.

Now, let F be a projective plane curve and P a simple point on F.

(2) Show that the tangent line to F at P has equation $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$.

Solution 2.

(1) From point 5) of Theorem 4.5, $I(P, F \cap L) \geq m_P(F) m_P(L) = m_P(F)$ with equality if and only if L is not a tangent line of F at P.

(2) Wlog assume $P \in \{X \neq 0\}$ and write $P = [1 : y_P : z_P]$. In $\{X \neq 0\}$, the tangent line has the equation

$$(F_*)_Y(P)(Y - y_P) + (F_*)_Z(P)(Z - Z_P) = 0,$$

and recall that $(F_*)_\bullet(P) = (F_\bullet)_*(P) = F_\bullet(P)$ (where $F_\bullet$ is any partial derivative). Also, by Euler's homogeneous function theorem, we have

$$F_X(P) + y_P F_Y(P) + z_P F_Z(P) = (\deg F) \cdot F(1, y_P, z_P) = 0$$

and hence we obtain the the equation of the tangent in $\{X \neq 0\}$ is

$$F_X(P) + F_Y(P)Y + F_X(P)Z = 0.$$

By homogenizing, we obtain, that the equation of the tangent in $\mathbb{P}^2$ is $F_X(P)X + F_Y(P)Y + F_X(P)Z = 0$.

Exercise 3. Show that the following projective plane curves are irreducible; find their multiple points and the tangents at multiple points with their multiplicities:

(1) $XY^4 + YZ^4 + XZ^4$
(2) $X^2Y^3 + X^2Z^3 + Y^2Z^3$
(3) $Y^2Z - X(X-Z)(X-\lambda Z)$, $\lambda \in k$
(4) $X^n + Y^n + Z^n$, $n > 0$

Solution 3. To distinguish between indeterminates and coordinates of points, we will use x, y, z for the coordinates of a point $P = [x : y : z]$.

(1) $(Y^4+Z^4)X + YZ^4$ is irreducible by Gauss' Lemma since it is irreducible in $k(Y,Z)[X]$ (as it is of degree 1 in X) and primitive in $k[Y,Z][X]$ (as Y, Z don't divide $Y^4 + Z^4$).

To determine the multiple points, we compute the partial derivatives:

$$F_X = Y^4 + Z^4, \quad F_Y = 4Y^3X + Z^4, \quad F_Z = 4Z^3Y + 4Z^3X.$$

If $P = [x : y : z]$, then $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$ implies $yz^4 = 0$ (combining $F(P) = 0$ and $F_X(P)$=0), so $y = 0$ or $z = 0$. But then $F_X(P) = 0$ gives $y = z = 0$ so $P = [1 : 0 : 0]$ is the only multiple point.

To determine the tangents at P, we work in $\{X \neq 0\}$, where P is just the origin. Dehomogenizing, we get

$$F_* = Y^4 + Z^4 + YZ^4.$$

If $\operatorname{char} k = 2$, then $Y^4 + Z^4 = (Y+Z)^4$, so F has the quadruple tangent $Y + Z$. If not, then there exists a primitive 8th root of unity ζ, and we have

$$Y^4 + Z^4 = (Y + \zeta Z)(Y - \zeta Z)(Y + i\zeta Z)(Y - i\zeta Z)$$

So in that case, F has the 4 distinct tangents $Y + \zeta Z$, $Y - \zeta Z$, $Y + i\zeta Z$, $Y - i\zeta Z$.

(2) $F = X^2(Y^3 + Z^3) + Y^2Z^3$ is irreducible since it is irreducible in $k(Z,Y)[X]$ ($-\frac{Z^3Y^2}{Z^3+Y^3}$ is not a square, and $Z^3 + Y^3$, Z^3Y^2 are coprime in $k[Y,Z]$).

To determine the multiple points, we compute the partial derivatives:

$$F_X = 2X(Y^3 + Z^3), \quad F_Y = Y(3X^2Y + 2Z^3), \quad F_Z = 3Z^2(X^2 + Y^2).$$

From the coefficients, it is clear that we should distinguish cases according to the characteristic:

char $k = 2$: From $F_Y = 0$ we obtain $xy = 0$, and then from $F_Z = 0$ we have $z(x+y) = 0$. This implies that precisely two of the coordinates are 0, and thus the multiple points are in $\{[1:0:0], [0:1:0], [0:0:1]\}$. It is starightforward to check that these are indeed multiple points.

char $k = 3$: From $F_Y = 0$ we obtain $yz = 0$, and $F_X = 0$ gives $x(y+z) = 0$. This then gives again the the solutions $\{[1:0:0], [0:1:0], [0:0:1]\}$, all of which are multiple points.

char $k \neq 2, 3$: If $Z \neq 0$, then $F_Z = 0$ gives $y^2 = -x^2$, and plugging this into $F = 0$ gives $xy = 0$, F_X gives $x = 0$, F_Y gives $y = 0$, so in this case we obtain $[0:0:1]$ which is indeed a multiple point. If $z = 0$, then F_Y gives $xy = 0$ so we obtain $[1:0:0]$ and $[0:1:0]$, which again are indeed multiple points.

In all three cases, the multiple points are precisely $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. Dehomogenizing with $X = 1$ gives that $[1:0:0]$ has multiplicity 3. The tangents are $Y + \zeta^i Z$ with $i \in \{0, 1, 2\}$ where ζ is a primitive 3rd root of unity if char $k \neq 3$ and $\zeta = 1$ if char $k = 3$. Dehomogenizing with $Y = 1$ gives that $[0:1:0]$ has multiplicity 2 with double tangent X. Dehomogenizing with $Z = 1$ gives that $[0:0:1]$ has multiplicity 2 with tangents $X \pm iY$, where i is a primitive 4th root of unity if char $k \neq 2$ and is 1 otherwise.

(3) $F = Y^2 Z - X(X-Z)(X-\lambda Z)$ irreducible since irreducible in $k(X,Z)[Y]$ and Z, $X(X-Z)(X-\lambda Z)$ coprime in $k[X,Z]$.

To obtain the multiple points, we compute the partial derivatives:

$$F_X = -(X-Z)(X-\lambda Z) - X(X-\lambda Z) - X(X-Z),$$
$$F_Y = 2YZ, \quad F_Z = Y^2 + (\lambda+1)X^2 - 2\lambda XZ.$$

Looking at F_Y it is clear that the case char $k = 2$ needs extra care.

char $k \neq 2$: In this case F_Y gives $yz = 0$. If $z = 0$, then F gives $x = 0$ and then F_Z gives $y = 0$, so we find no multiple points here. If $z \neq 0$, then we must have $y = 0$. Combining F and F_X, we see that x must be a double root of the polynomial $p(T) = T(T-z)(T-\lambda z)$ (as it annihilates both p and p'). If $\lambda \notin \{0, 1\}$, then p has no double root, so there are no multiple points. If $\lambda \in \{0, 1\}$, then p has the double root λz, so we obtain the candidate $[\lambda : 0 : 1]$ for the multiple point. It is straightforward to check that this is indeed a multiple point.

Now we determine the tangents: if $\lambda = 0$, the unique multiple point is $P = [0:0:1]$. Dehomogenizing with $Z = 1$ gives that the multiplicity is 2 and the tangents are $Y \pm X$. If $\lambda = 1$, the unique multiple point is $[1:0:1]$. Dehomogenizing with $Z = 1$ gives $F_* = Y^2 - X(X-1)^2$, and translating to $(1,0)$ gives

$$F_*(X+1,Y) = Y^2 - (X+1)X^2.$$

Hence the multiplicity is 2. The tangents here are $X \pm Y$, but we have to translate them back to $(1,0)$ giving $X \pm Y - 1$. These are now the tangents in $\{Z \neq 0\}$, so by homogenizing we obtain the the tangents are $X \pm Y - Z$.

$\operatorname{char} k = 2$: In this case F_Z gives $y^2 + x^2(\lambda+1) = 0$. If $x = 0$ then hence have $y = 0$ and so we obtain the candidate $[0:0:1]$. It is straightforward to check that this is a multiple point if and only if $\lambda = 0$. If $x \neq 0$, then plugging in $y^2 = (\lambda+1)x^2$ in the original equation and using that $\operatorname{char} k = 2$, we obtain $x^2 + \lambda z^2$. As every element in an algebraically closed field of characteristic 2 has a unique square root, we hence obtain that $x = \sqrt{\lambda} z$ and $y = \sqrt{\lambda} x$. This gives the candidate $[\sqrt{\lambda} : \sqrt{\lambda(\lambda+1)} : 1]$, and one can check that this is indeed a multiple point.

In conclusion, we obtain that for every λ, F has the unique multiple point $[\sqrt{\lambda} : \sqrt{\lambda(\lambda+1)} : 1]$. Let us compute the tangents: we dehomogenize with $Z = 1$, giving

$$F_* = Y^2 + X(X+1)(X+\lambda).$$

Translating to $(\sqrt{\lambda}, \sqrt{\lambda(\lambda+1)})$ then gives

$$F_*\left(X + \sqrt{\lambda}, Y + \sqrt{\lambda(\lambda+1)}\right) = Y^2 + (\lambda + \sqrt{\lambda} + 1)X^2 + X^3.$$

Hence the multiplicity is two, with double tangent $Y + (\lambda^{1/2} + \lambda^{1/4} + 1)X$. Translating back by replacing Y with $Y - \sqrt{\lambda(\lambda+1)}$ and X with $X - \sqrt{\lambda}$, we obtain that the equation of the double tangent is

$$Y + (\lambda^{1/2} + \lambda^{1/4} + 1)X + \lambda^{3/4}.$$

This is the equation in $\{Z \neq 0\}$, so to obtain the equation in $\mathbb{P}^2$ we have to dehomogenize, giving that

$$Y + (\lambda^{1/2} + \lambda^{1/4} + 1)X + \lambda^{3/4} Z$$

is the equation of the double tangent of F at its multiple point.

Remark. The curve F in this point of the exercise is what is called an elliptic curve, and in characteristic $\neq 2, 3$, all elliptic curves are isomorphic to F above with a specific choice of $\lambda \notin \{0,1\}$.

(4) If $p = \operatorname{char} k \mid n$, then F is reducible, because if we write $n = mp^r$ then

$$X^n + Y^n + Z^n = (X^m + Y^m + Z^m)^{p^r}.$$

So let us assume that $p \nmid n$. We want to show that F is irreducible by using Eisenstein's criterion in $k[Y,Z][X]$. For this it is enough to find an irreducible polynomial in $k[Y,Z]$ which divides $Y^n + Z^n$ but whose square doesn't. Notice that if $\zeta \in k$ is such that $\zeta^n = -1$, then $Y - \zeta Z$ is a factor of $Y^n + Z^n$. Let us write $Y^n + Z^n = (Y - \zeta Z)^l G(Y, Z)$ for some $G \in k[Y, Z]$. If we assume by contradiction that $l > 1$, then taking the derivative with respect to Y and plugging in $Y = \zeta Z$ gives that $n(\zeta Z)^{n-1} = 0$, contradiction (here we use $n \nmid \operatorname{char} k$). Hence $Y - \zeta Z$ is a simple factor of $Y^n + Z^n$, so by Eisenstein we obtain that $X^n + Y^n + Z^n$ is irreducible.

But then there are no multiple points: the partial derivatives of F are

$$F_X = nX^{n-1},\ F_Y = nY^{n-1},\ F_Z = nZ^{n-1},$$

and so they vanish simultaneously only for $x = y = z = 0$.

In the case where $n = mp^r$ with $r > 1$ and $p \nmid m$, then every point of F has multiplicity p^r, and the tangents are just p^r-fold multiples of the tangents of $X^m + Y^m + Z^m$.

Exercise 4. Find the intersection points and the intersection numbers of the following pairs of projective plane curves:

(1) $Y^2Z - X(X - 2Z)(X + Z)$ and $Y^2 + X^2 - 2XZ$
(2) $(X^2 + Y^2)Z + X^3 + Y^3$ and $X^3 + Y^3 - 2XYZ$
(3) $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$
(4) $(X^2 + Y^2)^2 + 3X^2YZ - Y^3Z$ and $(X^2 + Y^2)^3 - 4X^2Y^2Z^2$

Solution 4. We will always denote the first curve by F and the second curve by G.

(1) Replacing y^2 with $-x^2 + 2xz$ in the second equation gives $x \in \{0, \pm 2z\}$. If $x = 0$ then also $y = 0$ and thus we only obtain $P_1 = [0 : 0 : 1]$. If $x \neq 0$, then in particular we must have $\operatorname{char} k \neq 2$ and $x = \pm 2z$. Plugging this back into G, we obtain the three intersection points $P_2 = [2 : 0 : 1]$ and $P_{3,4} = [-2 : \pm\sqrt{-8} : 1]$.

For the intersection multiplicity, if $\operatorname{char} k = 2$, then as $P_1 = [0 : 0 : 1]$ is the only intersection point, we must have $I(P_1, F \cap G) = 6$ by Bezouts theorem. If $\operatorname{char} k \neq 2$, we have to compute the individual intersection multiplicities. Notice that all of the intersection points are in $\{Z \neq 0\}$, so we dehomogenize with $Z = 1$. By abuse of notation, we will still denote by P_i the corresponding point in the affine chart. We have

$$\begin{aligned} I(P_i, F \cap G) &= I(P_i, (Y^2 - X(X-2)(X+1)) \cap (Y^2 + X^2 - 2X)) \\ &\overset{7)}{=} I(P_i, X(X-2)(X+2) \cap (Y^2 + X^2 - 2X)) \\ &\overset{6)}{=} I(P_i, (X - P_{i,X}) \cap (Y^2 + X^2 - 2X)) \\ &\overset{7)}{=} I(P_i, (X - P_{i,X}) \cap (Y^2 - P_{i,Y}^2)), \end{aligned}$$

where we write $P_i = (P_{i,X}, P_{i,Y})$ (i.e. $P_i = [P_{i,X} : P_{i,Y} : 1]$). So if $P_{i,Y} = 0$ then $I(P_i, F \cap G) = 2$ and if $P_{i,Y} \neq 0$ then $I(P_i, F \cap G) = 1$. That is, at $P_{1,2}$, the curves intersect with multiplicity 2, and at $P_{3,4}$ they intersect with multiplicity 1.

(2) Replacing $x^3 + y^3$ in F with $2xyz$ gives $z(x+y)^2 = 0$. If $z = 0$, we obtain the intersection points $[1 : -\zeta^i : 0]$, where ζ is a primitive 3rd root of unity (if $\operatorname{char} k = 3$ we only have $[1 : -1 : 0]$). If $z \neq 0$, we obtain $x = -y$ and thus $2x^2 = 0$. If $\operatorname{char} k = 2$ then all points of the form $[x : -x : 1]$ are in the intersection, i.e. F and G have the common component $X + Y$. If $\operatorname{char} k \neq 2$, then we obtain only $[0 : 0 : 1]$.

We divide the computation of the intersection multiplicity according to the characteristic.

$\operatorname{char} k = 2$: In this case we have intersection multiplicity ∞ at points on $X + Y$. Apart from that, we only have the three intersection points $P_i = [1 : -\zeta^i : 0]$ for $i \in \{1, 2\}$. We dehomogenize with $X = 1$, giving

$$\begin{aligned} I(P_i, F \cap G) &\overset{7)}{=} I(P_i, ((1+Y)^2 Z) \cap (1 + Y^3)) \\ &\overset{6)+2)}{=} I(P_i, Z \cap (1 - Y + Y^2)) \\ &\overset{6)+2)}{=} I(P_i, Z \cap (Y - \zeta^i)) \\ &= 1. \end{aligned}$$

$\operatorname{char} k = 3$: In this case we obtained that the points of intersection are precisley $P_1 = [1 : -1 : 0]$ and $P_2 = [0 : 0 : 1]$. For P_1 we dehomogenize with $X = 1$ to obtain

$$\begin{aligned} I(P_1, F \cap G) &\overset{7)}{=} I(P_1, ((1+Y)^2 Z) \cap ((1+Y)^3 - 2YZ)) \\ &\overset{6)+7)}{=} 2I(P_1, (1+Y) \cap YZ) + I(P_1, Z \cap (1+Y)^3) \\ &\overset{6)+7)}{=} 2 + 3 \\ &= 5. \end{aligned}$$

By Bezout, we obtain that $I(P_2, F \cap G)$ is equal to $9 - 5 = 4$.

$\operatorname{char} k \neq 2, 3$: In this case we have the intersection points $P_i = [1 : -\zeta^i : 0]$ with $i \in \{0, 1, 2\}$ and $P_3 = [0 : 0 : 1]$. For $i \in \{0, 1, 2\}$ we dehomogenize with $X = 1$ and obtain

$$\begin{aligned} I(P_i, F \cap G) &\overset{7)}{=} I(P_i, ((1+Y)^2 Z) \cap (1 + Y^3 - 2YZ)) \\ &\overset{6)+7)}{=} 2I(P_i, (1+Y) \cap YZ) + I(P_i, Z \cap (1 + Y^3)) \\ &\overset{6)+2)}{=} 2I(P_i, (1+Y) \cap Z) + I(P_i, Z \cap (Y + \zeta^i)) \end{aligned}$$

For $i = 0$ we obtain 3, and for $i = 1, 2$ we obtain 1. By Bezout, this gives that $I(P_3, F \cap G) = 4$.

(3) Expanding $(y^2-xz)^2$ in F and replacing x^2z^2 with y^4+y^3z we obtain

$$y^2(y^3-x(2y^2-2xz+yz))=0.$$

If $y=0$, then G gives $xz=0$ so we obtain candidates $P_1=[1:0:0]$ and $P_2=[0:0:1]$, and it is straightforward to see that they are indeed intersection points. If $y\neq 0$ we may assume that $y=1$, and then from the above equation we obtain

$$\begin{aligned} &1-x(2-2xz+z)=0\\ \implies &(1-xz)(1-2x)=0. \end{aligned}$$

If $xz=1$ then G gives $z=0$, contradiction. So we obtain $1-2x=0$. So if $\operatorname{char} k=2$ then there are no further intersection points, and if $\operatorname{char} k\neq 2$, then G gives that z is a solution of $1+z-z^2/4$. If we denote these solutions by $z_{1,2}$, then we obtain candidates $P_{3,4}=[1/2:1:z_{1,2}]$, and it is straightforward to check that these are indeed intersection points.

To compute the intersection multiplicities, we distinguish cases along the characteristic.

$\operatorname{char} k=2$: In this case we only have P_1 and P_2. Let us compute the intersection multiplicity for P_2: we dehomogenize with $Z=1$ to obtain

$$\begin{aligned} I(P_2,F\cap G) &= I(P_2,(Y^5-X(Y^2-X)^2)\cap(Y^4+Y^3-X^2))\\ &\overset{7)}{=} I(P_2,(Y^5-XY^3)\cap(Y^4+Y^3-X^2))\\ &\overset{6)}{=} 3I(P_2,Y\cap(Y^4+Y^3-X^2))+I(P_2,(Y^2-X)\cap(Y^4+Y^3-X^2))\\ &\overset{6)}{=} 6+I(P_2,(Y^2-X)\cap(Y^4+Y^3-X^2))\\ &\overset{7)}{=} 6+I(P_2,(Y^2-X)\cap Y^3)\\ &\overset{6)+7)}{=} 9. \end{aligned}$$

By Bezout, we then obtain $I(P_1,F\cap G)=11$.

$\operatorname{char} k\neq 2$: We start with P_2, so we dehomogenize with $Z=1$ and obtain

$$\begin{aligned} I(P_2,F\cap G) &= I(P_2,(Y^5-X(Y^2-X)^2)\cap(Y^4+Y^3-X^2))\\ &\overset{7)}{=} I(P_2,(Y^5-X(Y^4-2XY^2+Y^4+Y^3))\cap G_*)\\ &\overset{6)}{=} 2\underbrace{I(P_2,Y\cap G_*)}_{=2}+I(P_2,\underbrace{(Y^3-X(2Y^2-2X+Y))}_{=(Y-2X)(Y^2-X)}\cap G_*)\\ &\overset{6)}{=} 4+\underbrace{I(P_2,(Y-2X)\cap G_*)}_{\overset{7)+6)+2)}{=}2}+\underbrace{I(P_2,(Y^2-X)\cap G_*)}_{\overset{7)+6)}{=}3}\\ &= 9. \end{aligned}$$

Similarly, for P_1 we dehomogenize with $X = 1$ and have

$$\begin{aligned}
I(P_1, F\cap G) &= I(P_1, (Y^5 - (Y^2 - Z)^2) \cap (Y^4 + Y^3Z - Z^2)) \\
&\overset{7)}{=} I(P_1, (Y^5 - (Y^4 - 2Y^2Z + Y^4 + Y^3Z)) \cap (Y^4 + Y^3Z - Z^2)) \\
&\overset{6)}{=} 2\underbrace{I(P_1, Y \cap (Y^4 + Y^3Z - Z^2))}_{=2} + I(P_1, \underbrace{(Y^3 - 2Y^2 - YZ + 2Z))}_{=(Y-2)(Y^2-Z)} \cap (Y^4 + Y^3Z - Z^2)) \\
&\overset{6)+2)}{=} 4 + \underbrace{I(P_1, (Y^2 - Z) \cap (Y^4 + Y^3Z - Z^2))}_{\overset{7)+6)}{=} 5} \\
&= 9.
\end{aligned}$$

By Bezout, this then implies also $I(P_{3,4}, F \cap G) = 1$.

(4) If $\operatorname{char} k = 2$, then G is $(X+Y)^6$ and F is $(X+Y)^2((X+Y)^2+YZ)$, so in this case every point $[x : -x : z]$ on G is also on F and we have intersection multiplicity ∞. So let us assume $\operatorname{char} k \neq 2$ from now on. If $xy = 0$ then G gives $x = y = 0$ so the candidate $P_1 = [0 : 0 : 1]$ which is indeed in the intersection. If $z = 0$ then we obtain $x^2+y^2 = 0$ which gives the candidates $P_{2,3} = [1 : \pm i : 0]$, which are indeed in the intersection. If $x^2+y^2 = 0$, then G gives $4x^2y^2z^2 = 0$ which is already covered by the above. Hence we may assume $x, y, z \neq 0$ and $x^2 + y^2 \neq 0$ from now on.

From F, we then obtain $y(3x^2-y^2) \neq 0$ and $z = \frac{(x^2+y^2)^2}{y(3x^2-y^2)}$. Plugging this into G we obtain

$$(x^2 + y^2)^3y^2(3x^2 - y^2)^2 - 4x^2y^2(x^2 + y^2)^4 = 0$$

and thus, as $y^2, x^2 + y^2 \neq 0$,

$$(3x^2 - y^2)^2 - 4x^2(x^2 + y^2) = 0.$$

Simplifying, we obtain

$$5x^4 - 10x^2y^2 + y^4 = 0.$$

As $x \neq 0$ we may assume $x = 1$. Notice that if $\operatorname{char} k = 5$, then $y = 0$ but this was already covered above. If $\operatorname{char} k \neq 5$, then as the discriminant of $\lambda^2-10\lambda+5 = 0$ is 80 and as we also suppose $\operatorname{char} k \neq 2$, we obtain that this quadratic equation has precisely two roots $\lambda_{1,2}$.We then have $y = \pm\sqrt{\lambda_{1,2}}$, and all of these values give us solutions to $F = G = 0$. Hence we obtain the 4 intersection points

$$P_{4,5,6,7} = \left[1 : \pm\sqrt{\lambda_{1,2}} : \pm\frac{(1 + \lambda_{1,2})^2}{\sqrt{\lambda_{1,2}}(\lambda_{1,2} - 3)}\right]$$

Notice that the intersection multiplicity at $[0 : 0 : 1]$ was computed in Exercise 3 on Sheet 10 (it is the multiplicity of $E \cap F$ at the origin): we have $I(P_1, F \cap G) = 18$ if $\operatorname{char} k = 5$ and 14 otherwise. Now let us compute $I(P_{2,3}, F \cap G)$: we dehomogenize with $X = 1$, giving

$$\begin{aligned}
I(P_{2,3}, F \cap G) &= I(P_{2,3}, ((1+Y^2)^2 + 3YZ - Y^3Z) \cap ((1+Y^2)^3 - 4Y^2Z^2)) \\
&\overset{7)}{=} I(P_{2,3}, ((1+Y^2)^2 + 3YZ - Y^3Z) \cap (-YZ(4YZ - (Y^2+1)(Y^2-3)))) \\
&\overset{6)+2)}{=} 2 + I(P_{2,3}, ((1+Y^2)^2 + 3YZ - Y^3Z) \cap (4YZ - (Y^2+1)(Y^2-3))) \\
&\overset{7)}{=} 2 + I(P_{2,3}, ((1+Y^2)(1+Y^2 - (Y^2-3)/4)) \cap (4YZ - (Y^2+1)(Y^2-3))) \\
&\overset{6)+2)}{=} 2 + I(P_{2,3}, (1+Y^2) \cap (4YZ - (Y^2+1)(Y^2-3)) \\
&\overset{7)}{=} 2 + I(P_{2,3}, (1+Y^2) \cap 4YZ) \\
&\overset{6)+2)}{=} 2 + I(P_{2,3}, (1+Y^2) \cap Z) \\
&= 3.
\end{aligned}$$

If $\operatorname{char} k = 5$ then with the multiplicity at P_1 this adds up to 24, confirming Bezouts theorem in this instance. If $\operatorname{char} k \neq 2, 5$, then the sum of multiplicities at $P_{1,2,3}$ is 20. By Bezouts theorem, we then obtain that the intersection multiplicity at $P_{4,5,6,7}$ must be equal to 1.